High-Precision Numerical Integration and Experimental Mathematics

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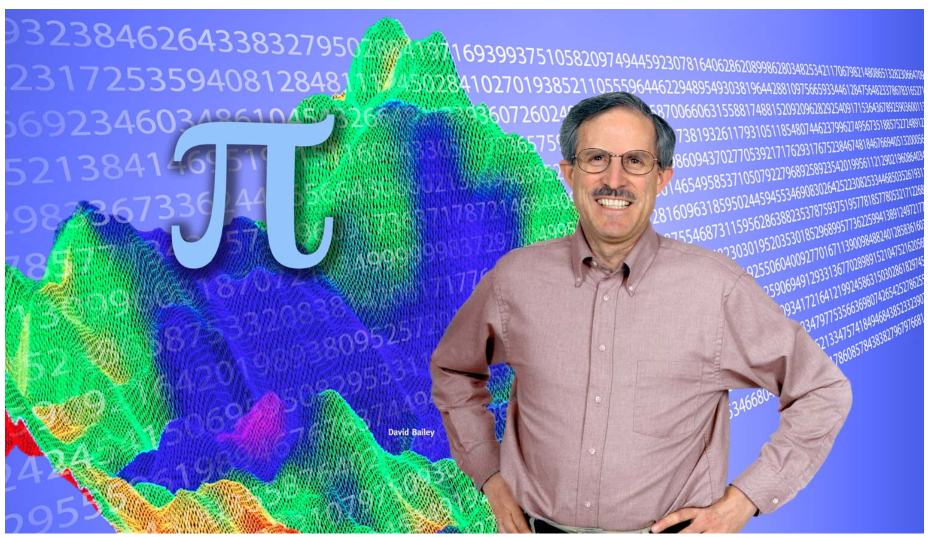


Available high-precision facilities

Vendor-supported arithmetic:

Type	Total Bits	Significant Digits	Support
IEEE Double	64	16	In hardware on almost all systems.
IEEE Extended	80	18	In hardware on Intel and AMD systems.
IEEE Quad	128	33	In software from some vendors (50-100X slower than IEEE double).

Non-commercial (free) software:

Type	Total Bits	Significant Digits	Support
Double-double	128	32	DDFUN90, QD.
Quad-double	256	64	QD.
Arbitrary	Any	Any	ARPREC, MPFUN90, GMP, MPFR.

Commercial software: *Mathematica, Maple*.

LBNL's high-precision software

- QD: double-double (31 digits) and quad-double (62 digits).
- ARPREC: arbitrary precision.
- Low-level routines written in C++.
- C++ and Fortran-90 translation modules permit use with existing C++ and Fortran-90 programs -- only minor code changes are required.
- Includes many common functions: sqrt, cos, exp, gamma, etc.
- PSLQ, root finding, numerical integration.

Available at: **http://www.experimentalmath.info**

Authors: Xiaoye Li, Yozo Hida, Brandon Thompson and DHB

High-precision arithmetic and experimental mathematics

"Experimental" methodology:

- Compute various mathematical entities (limits, infinite series sums, definite integrals) to high precision.
- Use algorithms such as PSLQ to recognize these entities in terms of well-known mathematical constants.
- Use this same process to discover relations between entities.
- When results are found experimentally, seek to find formal mathematical proofs of the discovered relations.
- Many results have been found using this methodology, both in pure math and in mathematical physics.

1. J. M. Borwein and DHB, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A.K. Peters, 2004. Second edition 2008.
2. J. M. Borwein, DHB and R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A.K. Peters, 2004.
3. DHB, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, V. Moll, *Experimental Mathematics in Action*, A.K. Peters, 2007.
4. J. M. Borwein and K. Devlin, *The Computer as Crucible: An Introduction to Experimental Mathematics*, A.K. Peters, 2007.

The PSLQ integer relation algorithm

Let (x_n) be a given vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

(or within "epsilon" of zero, where epsilon = 10^{-p} and p is the precision).

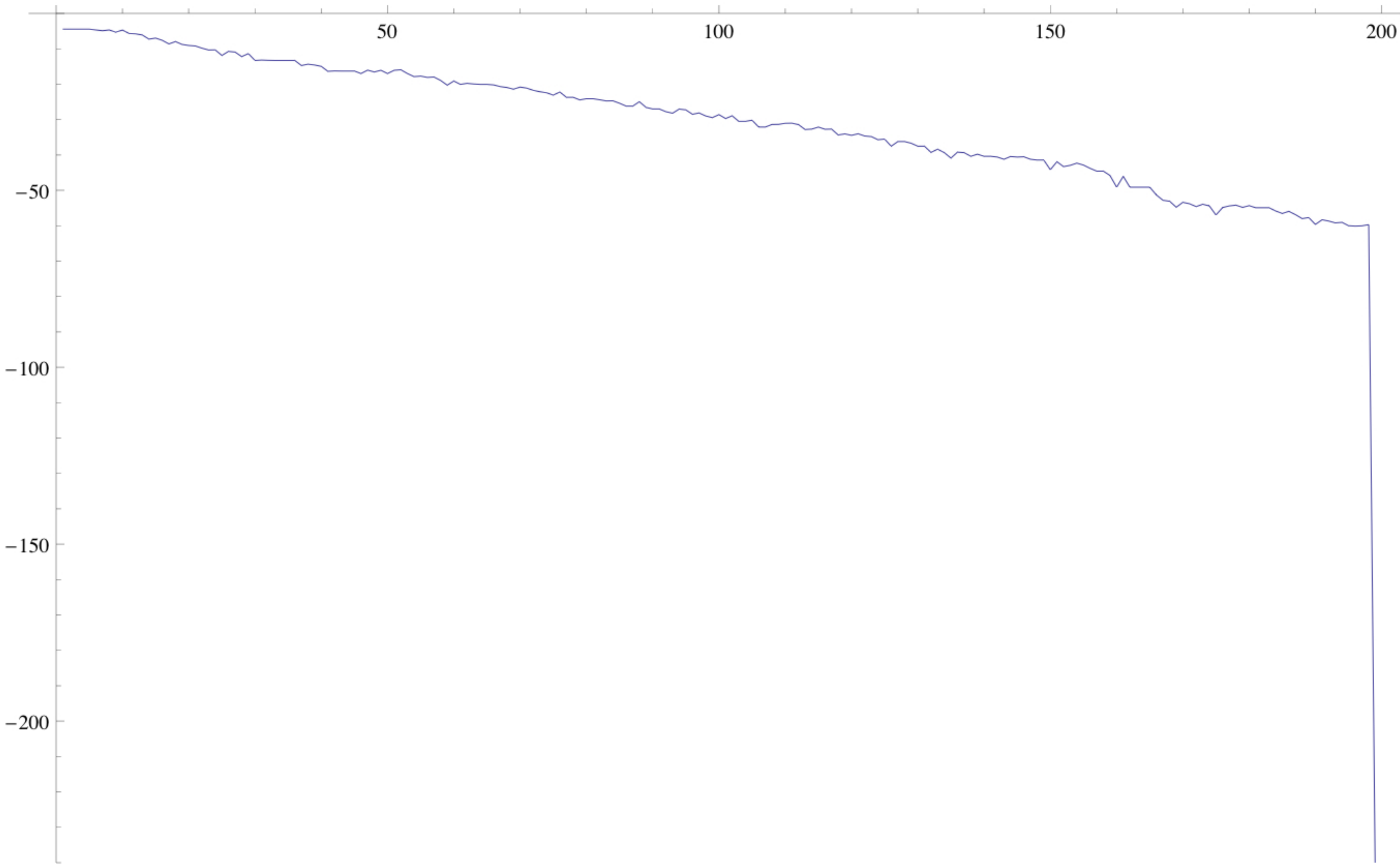
At the present time the "PSLQ" algorithm of mathematician-sculptor Helaman Ferguson is the most widely used integer relation algorithm. It was named one of ten "algorithms of the century" by *Computing in Science and Engineering*.

1. H. R. P. Ferguson, DHB and S. Arno, "Analysis of PSLQ, an integer relation finding algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), pg. 351-369.
2. DHB and D. J. Broadhurst, "Parallel integer relation detection: Techniques and applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), pg. 1719-1736.

PSLQ, continued

- PSLQ constructs a sequence of integer-valued matrices B_n that reduces the vector $y = x B_n$, until either the relation is found (as one of the columns of B_n), or else precision is exhausted.
- At the same time, PSLQ generates a steadily growing bound on the size of any possible relation.
- When a relation is found, the size of smallest entry of the vector y abruptly drops to roughly "epsilon" (i.e. 10^{-p}, where p is the number of digits of precision).
- The size of this drop can be viewed as a "confidence level" that the relation is real and not merely a numerical artifact -- a drop of 20+ orders of magnitude almost always indicates a real relation.
- PSLQ (or any other integer relation scheme) requires *very high precision arithmetic* (at least nd digits, where d is the size in digits of the largest a_k), both in the input data and in the operation of the algorithm.

Decrease of $\log_{10}(\min_k |y_k|)$ as a function of iteration number in a typical PSLQ run



Methodology for using PSLQ to recognize an unknown constant α

- Calculate α to high precision – typically 100 - 1000 digits. This is often the most computationally expensive part of the entire process.
- Based on experience with similar constants or relations, make a list of possible terms on the right-hand side (RHS) of a linear formula for α, then calculate each of the n RHS terms to the same precision as α.
- If you suspect α is algebraic of degree n (the root of a degree-n polynomial with integer coefficients), compute the vector $(1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n)$.
- Apply PSLQ to the (n+1)-long vector, using the same numeric precision as α, but with a detection threshold a few orders of magnitude larger than "epsilon"– e.g., 10^{-480} instead of 10^{-500} for 500-digit arithmetic.
- When PSLQ runs, look for a detection following a drop in the size of the reduced y vector by at least 20 orders of magnitude, to value near epsilon.
- If no credible relation is found, try expanding the list of RHS terms.
- Another possibility is to search for multiplicative relations (i.e., monomial expressions), which can be done by taking logarithms of α and constants.

History of numerical integration (quadrature)

- 1670: Newton devises the Newton-Coates integration rule.
- 1740: Thomas Simpson develops Simpson’s rule.
- 1820: Gauss develops Gaussian quadrature.
- 1950-1970: Adaptive quadrature, Romberg integration, Clenshaw-Curtis integration, others.
- 1973: Takashi and Mori develop the tanh-sinh quadrature scheme.
- 1985-1990: Maple and Mathematica feature built-in numerical quadrature facilities.
- 2000: Very high-precision quadrature (1000+ digits) methods.

With high-precision numerical values, we can now use PSLQ to obtain analytical evaluations of integrals.

Gaussian quadrature

Gaussian quadrature is often the most efficient scheme for regular functions (including at endpoints) and modest precision (< 1000 digits):

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \quad \approx \quad \sum_{j=1}^{n} w_j f(x_j)$$

The abscissas (x_j) are the roots of the n-th degree Legendre polynomial $P_n(x)$ on [-1,1]. The weights (w_j) are given by

$$w_j \quad = \quad \frac{-2}{(n+1)P_n'(x_j)P_{n+1}(x_j)}$$

The abscissas (x_j) are computed by Newton iterations, with starting values $\cos[\pi(j-1/4)/(n+1/2)]$. Legendre polynomials and their derivatives can be computed using the formulas $P_0(x) = 0$, $P_1(x) = 1$,

$$\begin{aligned}(k+1)P_{k+1}(x) &= (2k+1)xP_k(x) - kP_{k-1}(x) \\ P_n'(x) &= n(xP_n(x) - P_{n-1}(x))/(x^2-1)\end{aligned}$$

DHB, X.S. Li and K. Jeyabalan, "A comparison of three high-precision quadrature schemes," *Experimental Mathematics*, vol. 14 (2005), no. 3, pg 317-329.

Gaussian quadrature, continued

- In cases where the function and higher derivatives are completely regular even at the endpoints, Gaussian quadrature typically achieves quadratic convergence – doubling *n* approximately doubles the precision.
- In such cases, Gaussian quadrature is typcially faster than any other scheme (not including the cost of computing abscissa-weight pairs).
- If the function is not regular, accuracy results are typically rather poor.
- Abscissa-weight sets of size *n* are not related to those for any other *n*.
- The Newton iterations used to calculate abscissas can be performed with a numeric precision that nearly doubles with each successive iteration.
- The cost of computing abscissa-weight sets increases as the square of *n*. There is no known scheme for computing abscissa-weight sets that avoids this quadratic increase.
- For very high precision (>500 digits), the cost of computing abscissa-weight sets is often hundreds or thousands of times as expensive as computing the integral. But the abscissa-weight sets can be computed once and stored on a disk file.

The Euler-Maclaurin formula of numerical analysis

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= h\sum_{j=0}^{n} f(x_j) - \frac{h}{2}(f(a)+f(b)) \\ &\quad - \sum_{i=1}^{m} \frac{h^{2i}B_{2i}}{(2i)!}\left(D^{2i-1}f(b) - D^{2i-1}f(a)\right) - E(h) \\ |E(h)| &\leq 2(b-1)(h/(2\pi))^{2m+2} \max_{a\leq x\leq b} |D^{2m+2}f(x)|\end{aligned}$$

Here $h = (b - a)/n$ and $x_j = a + jh$; B_{2i} are Bernoulli numbers; $D^m f(x)$ is the m-th derivative of $f(x)$.

Note when $f(x)$ and all of its derivatives are zero at the endpoints a and b (as in a bell-shaped curve), the error $E(h)$ of a simple trapezoidal approximation to the integral goes to zero more rapidly than any power of h.

K. Atkinson, *An Introduction to Numerical Analysis*, John Wiley, 1989, pg. 289.

Tanh-sinh quadrature

Given $f(x)$ defined on (-1,1), define $g(t) = \tanh(\pi/2 \sinh t)$. Then setting $x = g(t)$ yields

$$\int_{-1}^{1} f(x)\,dx = \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt \approx h \sum_{j=-N}^{N} w_j f(x_j),$$

where $x_j = g(hj)$ and $w_j = g'(hj)$. Since $g'(t)$ goes to zero very rapidly for large t, the product $f(g(t))\, g'(t)$ typically is a nice bell-shaped function, and thus the simple summation above is a remarkably accurate. The summation is continued until the terms are negligible.

The abscissas and weights are computed by these formulas:

$$\begin{aligned} x_j &= \tanh[\pi/2 \cdot \sinh(hj)] \\ w_j &= \pi/2 \cdot \cosh(hj) \cosh^2[\pi/2 \cdot \sinh(hj)] \end{aligned}$$

1. DHB, X.S. Li and K. Jeyabalan, “A comparison of three high-precision quadrature schemes,” *Experimental Mathematics*, vol. 14 (2005), no. 3, pg. 317-329.
2. H. Takahasi and M. Mori, “Double exponential formulas for numerical integration,” Publications of RIMS, Kyoto University, vol. 9 (1974), pg. 721–741.

Tanh-sinh quadrature, continued

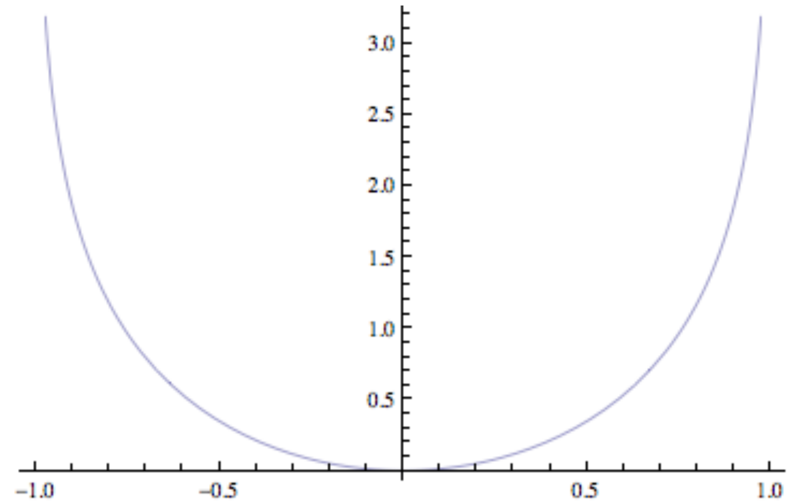
- The requisite conditions are met by a wide range of “reasonable” integrand functions, even functions with vertical derivatives or blow-up singularities at the endpoints.
- In some cases, even functions that appear regular may have singularities at higher-level derivatives; Gaussian quadrature cannot handle such functions, but tanh-sinh can.
- For “good” functions, reducing h by half (i.e., doubling the number of abscissa-weight pairs) approximately doubles the result precision.
- Abscissa-weight pairs should be generated up to N such that the weight is approximately 10^{-2p}, where p is the precision level in digits.
- The set of abscissa-weight pairs for a given N are merely the even-indexed pairs for the abscissa-weight set of size $2N$.
- The cost of computing abscissa-weight sets increases only linearly with N.
- For very high precision (>500 digits), tanh-sinh is often the algorithm of choice even for regular functions, due to its much lower cost for computing abscissa-weight sets.

Original and transformed integrand functions

Original integrand function on [-1,1]:

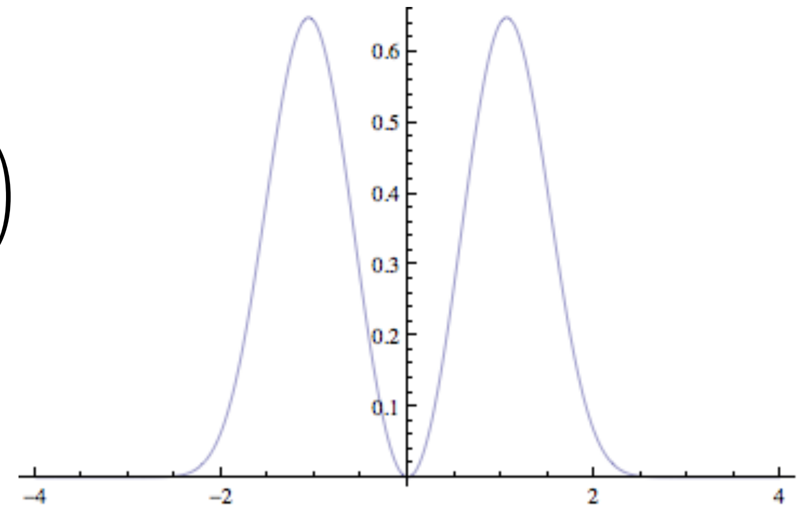
$$f(x) = -\log\cos\left(\frac{\pi x}{2}\right)$$

Note the singularities at the endpoints.



Transformed using *x* = *g*(*t*) = tanh (sinh *t*):

$$f(g(t))g'(t) \quad = \quad -\log\cos[\pi/2\cdot\tanh(\sinh t)]\left(\frac{\cosh(t)}{\cosh(\sinh t)^2}\right)$$



This is now a nice smooth bell-shaped function, so the E-M formula implies that a trapezoidal approximation is very accurate.

Erf quadrature

Erf quadrature is similar to tanh-sinh quadrature:

$$\int_{-1}^{1} f(x)\,dx \quad = \quad \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt \;\approx\; h \sum_{j=-N}^{N} w_j f(x_j),$$

where

$$\begin{aligned} g(t) &= \operatorname{erf}(t) \\ x_j &= \operatorname{erf}(hj) \\ w_j &= (2/\sqrt{\pi})e^{-(hj)^2} \\ \operatorname{erf}(x) &= (2/\sqrt{\pi}) \int_0^x e^{-t^2}\,\mathrm{d}t \end{aligned}$$

The erf function can be calculated to high precision using known formulas.

A suite of test integrals

Continuous functions on closed intervals:

$$1 \quad : \quad \int_0^1 t \log(1+t)\, dt \;=\; 1/4$$

$$2 \quad : \quad \int_0^1 t^2 \arctan t\, dt \;=\; (\pi - 2 + 2\log 2)/12$$

$$3 \quad : \quad \int_0^{\pi/2} e^t \cos t\, dt \;=\; (e^{\pi/2} - 1)/2$$

$$4 \quad : \quad \int_0^1 \frac{\arctan(\sqrt{2+t^2})}{(1+t^2)\sqrt{2+t^2}}\, dt \;=\; 5\pi^2/96$$

A suite of test integrals, continued

Continuous functions with infinite derivative at an endpoint:

$$5 : \int_0^1 \sqrt{t}\log t\,dt = -4/9$$

$$6 : \int_0^1 \sqrt{1-t^2}\,dt = \pi/4$$

Continuous functions with a blow-up singularity at an endpoint:

$$7 : \int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}dt = 2\sqrt{\pi}\Gamma(3/4)/\Gamma(1/4)$$

$$8 : \int_0^1 \log^2 t\,dt = 2$$

$$9 : \int_0^{\pi/2} \log(\cos t)\,dt = -\pi\log(2)/2$$

$$10 : \int_0^{\pi/2} \sqrt{\tan t}\,dt = \pi\sqrt{2}/2$$

A suite of test integrals, continued

Continuous functions on an infinite interval:

$$11 \quad : \quad \int_0^\infty \frac{1}{1+t^2}\,dt \;=\; \int_0^1 \frac{ds}{1-2s+2s^2} \;=\; \pi/2$$

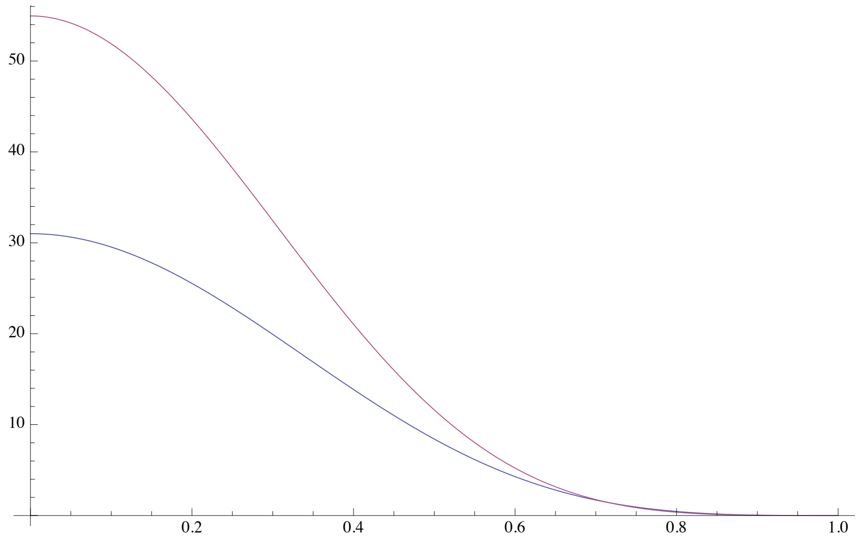
$$12 \quad : \quad \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt \;=\; \int_0^1 \frac{e^{1-1/s}\,ds}{\sqrt{s^3-s^4}} \;=\; \sqrt{\pi}$$

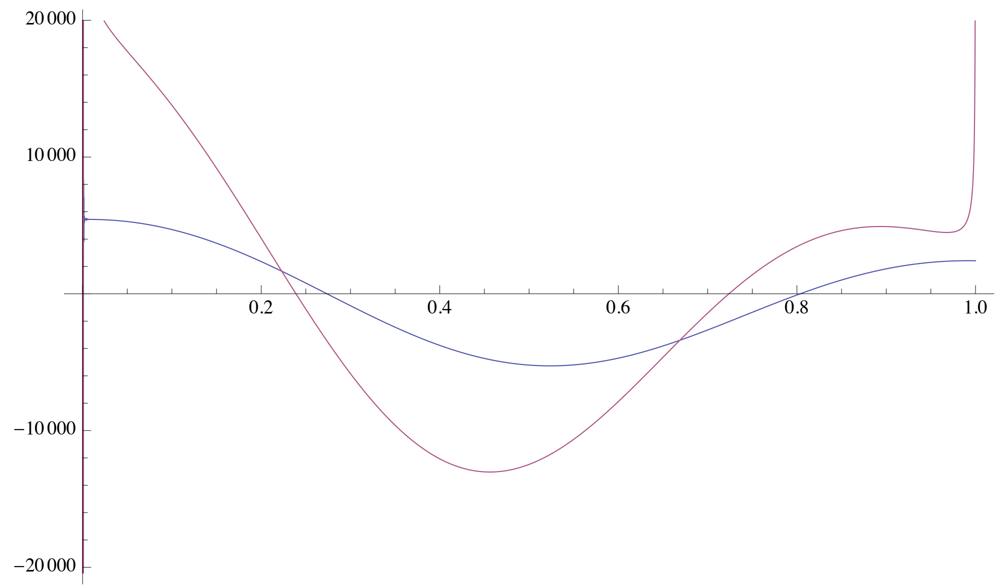
$$13 \quad : \quad \int_0^\infty e^{-t^2/2}\,dt \;=\; \int_0^1 \frac{e^{-(1/s-1)^2/2}\,ds}{s^2} \;=\; \sqrt{\pi/2}$$

Comparison of three quadrature schemes

	QUADGS			QUADERF			QUADTS		
Prob.	Level	Time	Error	Level	Time	Error	Level	Time	Error
Init	12	73,046.28		13	3,891.63		12	390.83	
1	7	6.86	10^{-1012}	10	97.16	10^{-1004}	9	37.33	10^{-1010}
2	7	9.13	10^{-1011}	11	112.11	10^{-1003}	9	32.64	10^{-1010}
3	7	10.01	10^{-1009}	10	90.29	10^{-1004}	9	41.23	10^{-1008}
4	7	9.31	10^{-1010}	11	453.92	10^{-1003}	9	67.39	10^{-1009}
5	12	14.70	10^{-13}	10	88.43	10^{-1004}	8	18.54	10^{-1010}
6	12	1.39	10^{-15}	10	6.75	10^{-1004}	9	2.29	10^{-1010}
7	12	2.49	10^{-5}	10	15.21	10^{-1001}	9	4.40	10^{-1002}
8	12	13.89	10^{-8}	10	98.25	10^{-1004}	8	19.19	10^{-1009}
9	12	18.66	10^{-9}	10	113.49	10^{-1004}	9	48.18	10^{-1008}
10	12	7.06	10^{-5}	10	35.80	10^{-1001}	9	15.55	10^{-1002}
11	8	0.41	10^{-1012}	11	10.41	10^{-1003}	10	3.03	10^{-1009}
12	12	7.98	10^{-5}	13	211.03	10^{-1001}	11	65.05	10^{-1002}
13	11	98.50	10^{-1011}	13	117.09	10^{-1003}	12	85.61	10^{-1007}

A “nice” function that requires tanh-sinh





Plots of

$$f(x) \quad = \quad \sin^p(\pi x)\zeta(p, x)$$

and its fourth derivative, for p = 3 (blue) and p = 3.5 (red).

In the case p = 3.5, because of the singularity in the fourth derivative, Gaussian quadrature gives very poor results; but tanh-sinh works fine.

A log-tan integral identity

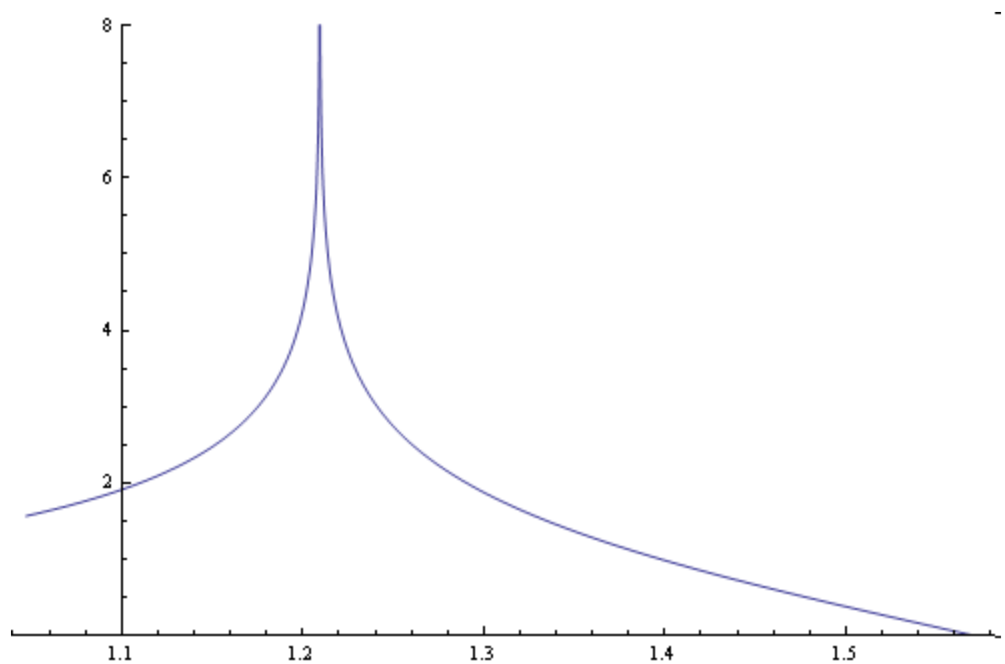
$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| \mathrm{d}t \quad = \quad L_{-7}(2) \quad =$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]$$

This identity arises from analysis of volumes of knot complements in hyperbolic space. This is simplest of 998 related identities.

We verified this numerically to 20,000 digits (using highly parallel tanh-sinh quadrature). A proof is now known.

DHB, J. M. Borwein, V. Kapoor and E. Weisstein, "Ten problems in experimental mathematics," *American Mathematical Monthly*, vol. 113, no. 6 (Jun 2006), pg. 481-409 .



Computing high-precision values of multi-dimension integrals

Computing multi-hundred digit numerical values of 2-D, 3-D and higher-dimensional integrals remains a major challenge.

Typical approach:

- Consider the 2-D or 3-D domain divided into 1-D lines.
- Use Gaussian quadrature (for regular functions) or tanh-sinh quadrature (if function has vertical derivates or singularities on boundaries) on each of the 1-D lines.
- Discontinue evaluation beyond points where it is clear that function-weight products are smaller than the “epsilon” of the precision level (this works better with tanh-sinh).

Even with “smart” evaluation that avoids unnecessary evaluations, the computational cost increases very sharply with dimension:

- If 1000 evaluation points are required in 1-D for a given precision, then typically 1,000,000 are required in 2-D and 1,000,000,000 in 3-D, etc.

Ising integrals

We recently applied our methods to study three classes of integrals that arise in the Ising theory of mathematical physics – D_n and two others:

$$C_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$D_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i - u_j}{u_i + u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$E_n = 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n} \frac{u_k - u_j}{u_k + u_j}\right)^2 dt_2\, dt_3 \cdots dt_n$$

where in the last line $u_k = t_1\, t_2 \ldots t_k$.

DHB, J. M. Borwein and R. E. Crandall, “Integrals of the Ising class,” *Journal of Physics A: Mathematical and General*, vol. 39 (2006), pg. 12271-12302.

Computing and evaluating C_n

We observed that the multi-dimensional C_n integrals can be transformed to 1-D integrals:

$$C_n = \frac{2^n}{n!} \int_0^\infty t K_0^n(t)\, \mathrm{d}t$$

where K_0 is the modified Bessel function. In this form, the C_n constants appear naturally in in quantum field theory (QFT).

We used this formula to compute 1000-digit numerical values of various C_n, from which the following results and others were found, then proven:

$$\begin{aligned} C_1 &= 2 \\ C_2 &= 1 \\ C_3 &= \mathrm{L}_{-3}(2) = \sum_{n \geq 0} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right) \\ C_4 &= \frac{7}{12} \zeta(3) \end{aligned}$$

Limiting value of C_n

The C_n numerical values appear to approach a limit. For instance,

$$C_{1024} = 0.63047350337438679612204019271087890435458707871273234\ldots$$

What is this limit? We copied the first 50 digits of this numerical value into the online Inverse Symbolic Calculator (ISC):

http://ddrive.cs.dal.ca/~isc

The result was:

$$\lim_{n \to \infty} C_n \quad = \quad 2e^{-2\gamma}$$

where gamma denotes Euler's constant. Finding this limit led us to the asymptotic expansion and made it clear that the integral representation of C_n is fundamental.

Other Ising integral evaluations

$$
\begin{aligned}
D_2 &= 1/3 \\
D_3 &= 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2) \\
D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2 \\
E_2 &= 6 - 8\log 2 \\
E_3 &= 10 - 2\pi^2 - 8\log 2 + 32\log^2 2 \\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 \\
&\quad +16\pi^2\log 2 - 22\pi^2/3 \\
E_5 &\stackrel{?}{=} 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 \\
&\quad +40\pi^2\log^2 2 - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 \\
&\quad +464\log^2 2 - 40\log 2
\end{aligned}
$$

where $\mathrm{Li}_n(x)$ is the polylog function. D_2, D_3 and D_4 were originally provided to us by mathematical physicist Craig Tracy, who hoped that our tools could help identify D_5.

The Ising integral E_5

We were able to reduce E_5, which is a 5-D integral, to an extremely complicated 3-D integral.

We computed this integral to 250-digit precision, using a highly parallel, high-precision 3-D quadrature program. Then we used a PSLQ program to discover the evaluation given on the previous page.

We also computed D_5 to 500 digits, but were unable to identify it. The digits are available if anyone wishes to further explore this question.

$$
\begin{aligned}
E_5 \;=\; & \int_0^1 \int_0^1 \int_0^1 [2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \\
& (-[4(x+1)(xy+1)\log(2)(y^5z^3x^7 - y^4z^2(4(y+1)z+3)x^6 - y^3z((y^2+1)z^2+4(y+ \\
& 1)z+5)x^5 + y^2(4y(y+1)z^3+3(y^2+1)z^2+4(y+1)z-1)x^4 + y(z(z^2+4z \\
& +5)y^2+4(z^2+1)y+5z+4)x^3 + ((-3z^2-4z+1)y^2-4zy+1)x^2 - (y(5z+4) \\
& +4)x-1)] / [(x-1)^3(xy-1)^3(xyz-1)^3] + [3(y-1)^2y^4(z-1)^2z^2(yz \\
& -1)^2x^6 + 2y^3z(3(z-1)^2z^3y^5 + z^2(5z^3+3z^2+3z+5)y^4 + (z-1)^2z \\
& (5z^2+16z+5)y^3 + (3z^5+3z^4-22z^3-22z^2+3z+3)y^2 + 3(-2z^4+z^3+2 \\
& z^2+z-2)y+3z^3+5z^2+5z+3)x^5 + y^2(7(z-1)^2z^4y^6 - 2z^3(z^3+15z^2 \\
& +15z+1)y^5 + 2z^2(-21z^4+6z^3+14z^2+6z-21)y^4 - 2z(z^5-6z^4-27z^3 \\
& -27z^2-6z+1)y^3 + (7z^6-30z^5+28z^4+54z^3+28z^2-30z+7)y^2 - 2(7z^5 \\
& +15z^4-6z^3-6z^2+15z+7)y + 7z^4-2z^3-42z^2-2z+7)x^4 - 2y(z^3(z^3 \\
& -9z^2-9z+1)y^6 + z^2(7z^4-14z^3-18z^2-14z+7)y^5 + z(7z^5+14z^4+3 \\
& z^3+3z^2+14z+7)y^4 + (z^6-14z^5+3z^4+84z^3+3z^2-14z+1)y^3 - 3(3z^5 \\
& +6z^4-z^3-z^2+6z+3)y^2 - (9z^4+14z^3-14z^2+14z+9)y + z^3+7z^2+7z \\
& +1)x^3 + (z^2(11z^4+6z^3-66z^2+6z+11)y^6 + 2z(5z^5+13z^4-2z^3-2z^2 \\
& +13z+5)y^5 + (11z^6+26z^5+44z^4-66z^3+44z^2+26z+11)y^4 + (6z^5-4 \\
& z^4-66z^3-66z^2-4z+6)y^3 - 2(33z^4+2z^3-22z^2+2z+33)y^2 + (6z^3+26 \\
& z^2+26z+6)y + 11z^2+10z+11)x^2 - 2(z^2(5z^3+3z^2+3z+5)y^5 + z(22z^4 \\
& +5z^3-22z^2+5z+22)y^4 + (5z^5+5z^4-26z^3-26z^2+5z+5)y^3 + (3z^4- \\
& 22z^3-26z^2-22z+3)y^2 + (3z^3+5z^2+5z+3)y + 5z^2+22z+5)x + 15z^2+2z \\
& +2y(z-1)^2(z+1) + 2y^3(z-1)^2z(z+1) + y^4z^2(15z^2+2z+15) + y^2(15z^4 \\
& -2z^3-90z^2-2z+15)+15] / [(x-1)^2(y-1)^2(xy-1)^2(z-1)^2(yz-1)^2 \\
& (xyz-1)^2] - [4(x+1)(y+1)(yz+1)(-z^2y^4+4z(z+1)y^3+(z^2+1)y^2 \\
& -4(z+1)y+4x(y^2-1)(y^2z^2-1) + x^2(z^2y^4-4z(z+1)y^3-(z^2+1)y^2 \\
& +4(z+1)y+1)-1)\log(x+1)] / [(x-1)^3x(y-1)^3(yz-1)^3] - [4(y+1)(xy \\
& +1)(z+1)(x^2(z^2-4z-1)y^4 + 4x(x+1)(z^2-1)y^3 - (x^2+1)(z^2-4z-1) \\
& y^2 - 4(x+1)(z^2-1)y + z^2-4z-1)\log(xy+1)] / [x(y-1)^3y(xy-1)^3(z- \\
& 1)^3] - [4(z+1)(yz+1)(x^3y^5z^7 + x^2y^4(4x(y+1)+5)z^6 - xy^3((y^2+ \\
& 1)x^2-4(y+1)x-3)z^5 - y^2(4y(y+1)x^3+5(y^2+1)x^2+4(y+1)x+1)z^4+ \\
& y(y^2x^3-4y(y+1)x^2-3(y^2+1)x-4(y+1))z^3 + (5x^2y^2+y^2+4x(y+1) \\
& y+1)z^2 + ((3x+4)y+4)z-1)\log(xyz+1)] / [xy(z-1)^3z(yz-1)^3(xyz-1)^3])] \\
& / [(x+1)^2(y+1)^2(xy+1)^2(z+1)^2(yz+1)^2(xyz+1)^2] \; dx\,dy\,dz
\end{aligned}
$$

Recursions in Ising integrals

Consider the 2-parameter class of Ising integrals (which arises in QFT for odd *k*):

$$C_{n,k} = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^{k+1}} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

After computing 1000-digit numerical values for all *n* up to 36 and all *k* up to 75 (performed on a highly parallel computer system), we discovered (using PSLQ) linear relations in the rows of this array. For example, when *n* = 3:

$$\begin{aligned}
0 &= C_{3,0} - 84C_{3,2} + 216C_{3,4} \\
0 &= 2C_{3,1} - 69C_{3,3} + 135C_{3,5} \\
0 &= C_{3,2} - 24C_{3,4} + 40C_{3,6} \\
0 &= 32C_{3,3} - 630C_{3,5} + 945C_{3,7} \\
0 &= 125C_{3,4} - 2172C_{3,6} + 3024C_{3,8}
\end{aligned}$$

Similar, but more complicated, recursions have been found for all *n*.

DHB, D. Borwein, J.M. Borwein and R.E. Crandall, “Hypergeometric forms for Ising-class integrals,” *Experimental Mathematics*, vol. 16 (2007), pg. 257-276.

J. M. Borwein and B. Salvy, “A proof of a recursion for Bessel moments,” *Experimental Mathematics*, vol. 17 (2008), pg. 223-230.

Four hypergeometric evaluations

$$c_{3,0} = \frac{3\Gamma^6(1/3)}{32\pi 2^{2/3}} = \frac{\sqrt{3}\pi^3}{8}\, {}_3F_2\left(\begin{array}{c} 1/2, 1/2, 1/2 \\ 1, 1 \end{array} \middle| \frac{1}{4}\right)$$

$$c_{3,2} = \frac{\sqrt{3}\pi^3}{288}\, {}_3F_2\left(\begin{array}{c} 1/2, 1/2, 1/2 \\ 2, 2 \end{array} \middle| \frac{1}{4}\right)$$

$$c_{4,0} = \frac{\pi^4}{4} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4}{4^{4n}} = \frac{\pi^4}{4}\, {}_4F_3\left(\begin{array}{c} 1/2, 1/2, 1/2, 1/2 \\ 1, 1, 1 \end{array} \middle| 1\right)$$

$$c_{4,2} = \frac{\pi^4}{64}\left[4\, {}_4F_3\left(\begin{array}{c} 1/2, 1/2, 1/2, 1/2 \\ 1, 1, 1 \end{array} \middle| 1\right) - 3\, {}_4F_3\left(\begin{array}{c} 1/2, 1/2, 1/2, 1/2 \\ 2, 1, 1 \end{array} \middle| 1\right)\right] - \frac{3\pi^2}{16}$$

DHB, J.M. Borwein, D.M. Broadhurst and M.L. Glasser, “Elliptic integral representation of Bessel moments,” *Journal of Physics A: Mathematical and Theoretical*, vol. 41 (2008), 5203-5231.

2-D integral in Bessel moment study

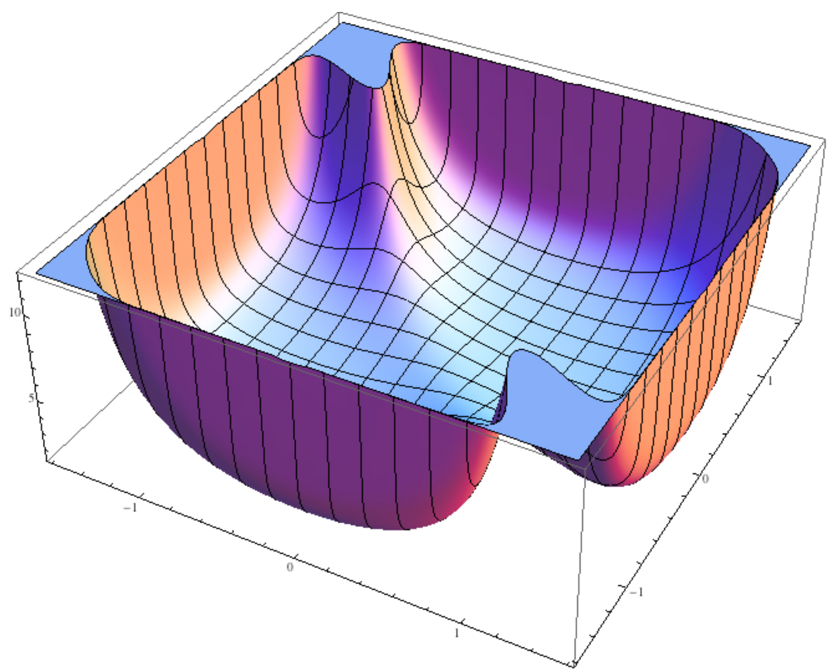
We conjectured (and later proved)

$$c_{5,0} = \frac{\pi}{2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{\mathbf{K}(\sin\theta)\,\mathbf{K}(\sin\phi)}{\sqrt{\cos^2\theta\cos^2\phi+4\sin^2(\theta+\phi)}}\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Here **K** denotes the complete elliptic integral of the first kind

Note that the integrand function has singularities on all four sides of the region of integration.

We were able to evaluate this integral to 120-digit accuracy, using 1024 cores of the "Franklin" Cray XT4 system at LBNL.



Heisenberg spin integrals

In another recent application of these methods, we investigated the following "spin integrals," which arise from studies in mathematical physics:

$$P(n) \quad := \quad \frac{\pi^{n(n+1)/2}}{(2\pi i)^n} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} U(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) \times T(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) \, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

where

$$U(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) = \frac{\prod_{1 \le k < j \le n} \sinh[\pi(x_j - x_k)]}{\prod_{1 \le j \le n} i^n \cosh^n(\pi x_j)}$$

$$T(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) = \frac{\prod_{1 \le j \le n} (x_j - i/2)^{j-1} (x_j + i/2)^{n-j}}{\prod_{1 \le k < j \le n} (x_j - x_k - i)}$$

H. E. Boos, V. E. Korepin, Y. Nishiyama and M. Shiroishi, "Quantum correlations and number theory," *Journal of Physics A: Mathematical and General*, vol. 35 (2002), pg. 4443.

Evaluations of *P*(*n*)
Derived analytically, confirmed numerically

$$P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{3} - \frac{1}{3}\log 2, \quad P(3) = \frac{1}{4} - \log 2 + \frac{3}{8}\zeta(3)$$

$$P(4) = \frac{1}{5} - 2\log 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\zeta(3)\log 2 - \frac{51}{80}\zeta^2(3) - \frac{55}{24}\zeta(5) + \frac{85}{24}\zeta(5)\log 2$$

$$\begin{aligned} P(5) = {} & \frac{1}{6} - \frac{10}{3}\log 2 + \frac{281}{24}\zeta(3) - \frac{45}{2}\zeta(3)\log 2 - \frac{489}{16}\zeta^2(3) - \frac{6775}{192}\zeta(5) \\ & + \frac{1225}{6}\zeta(5)\log 2 - \frac{425}{64}\zeta(3)\zeta(5) - \frac{12125}{256}\zeta^2(5) + \frac{6223}{256}\zeta(7) \\ & - \frac{11515}{64}\zeta(7)\log 2 + \frac{42777}{512}\zeta(3)\zeta(7) \end{aligned}$$

and a much more complicated expression for *P*(6). Run times increase very rapidly with the dimension *n*:

n	Digits	Processors	Run Time
2	120	1	10 sec.
3	120	8	55 min.
4	60	64	27 min.
5	30	256	39 min.
6	6	256	59 hrs.

Box integrals

The following integrals appear in numerous arenas of math and physics:

$$B_n(s) := \int_0^1 \cdots \int_0^1 \left(r_1^2 + \cdots + r_n^2\right)^{s/2} \, dr_1 \cdots dr_n$$

$$\Delta_n(s) := \int_0^1 \cdots \int_0^1 \left((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2\right)^{s/2} \, dr_1 \cdots dr_n \, dq_1 \cdots dq_n$$

- $B_n(1)$ is the expected distance of a random point from the origin of n-cube.
- $\Delta_n(1)$ is the expected distance between two random points in n-cube.
- $B_n(-n+2)$ is the expected electrostatic potential in an n-cube whose origin has a unit charge.
- $\Delta_n(-n+2)$ is the expected electrostatic energy between two points in a uniform n-cube of charged “jellium.”
- Recently integrals of this type have arisen in neuroscience – e.g., the average distance between synapses in a mouse brain.

DHB, J. M. Borwein and R. E. Crandall, “Box integrals,” *Journal of Computational and Applied Mathematics*, vol. 206 (2007), pg. 196-208.

Recent result (18 Jan 2009)

$$
\begin{aligned}
\Delta_3(-1) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{(-1 + e^{-u^2} + \sqrt{\pi}\, u\, \mathrm{erf}(u))^3}{u^6}\, \mathrm{d}u \\
&= \frac{1}{15}\left(6 + 6\sqrt{2} - 12\sqrt{3} - 10\pi + 30\log(1+\sqrt{2}) + 30\log(2+\sqrt{3})\right)
\end{aligned}
$$

As in many of the previous results, this was found by first computing the integral to high precision (250 to 1000 digits), conjecturing possible terms on the right-hand side, then applying PSLQ to look for a relation. We now have proven this result.

Dozens of similar results have since been found (see next few viewgraphs), raising hope that all box integrals eventually will be evaluated in closed form.

DHB, J. M. Borwein and R. E. Crandall, “Advances in the theory of box integrals,” manuscript, Mar 2009, available at http://crd.lbl.gov/~dhbailey/dhbpapers/BoxII.pdf.

Recent evaluations of box integrals

n	s	$B_n(s)$
any	even $s \geq 0$	rational, e.g., : $B_2(2) = 2/3$
1	$s \neq -1$	$\frac{1}{s+1}$
2	-4	$-\frac{1}{4} - \frac{\pi}{8}$
2	-3	$-\sqrt{2}$
2	-1	$2\log(1+\sqrt{2})$
2	1	$\frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2})$
2	3	$\frac{7}{5}\sqrt{2} + \frac{3}{20}\log(1+\sqrt{2})$
2	$s \neq -2$	$\frac{2}{2+s}\ {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right)$
3	-5	$-\frac{1}{6}\sqrt{3} - \frac{1}{12}\pi$
3	-4	$-\frac{3}{2}\sqrt{2}\arctan\frac{1}{\sqrt{2}}$
3	-2	$-3\mathrm{G} + \frac{3}{2}\pi\log(1+\sqrt{2}) + 3\ \mathrm{Ti}_2(3-2\sqrt{2})$
3	-1	$-\frac{1}{4}\pi + \frac{3}{2}\log\left(2+\sqrt{3}\right)$
3	1	$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right)$
3	3	$\frac{2}{5}\sqrt{3} - \frac{1}{60}\pi - \frac{7}{20}\log\left(2+\sqrt{3}\right)$

Here *F* is hypergeometric function; *G* is Catalan; Ti is Lewin's inverse-tan function.

Recent evaluations of box integrals, continued

n	s	$B_n(s)$
4	-5	$-\sqrt{8}\ \arctan\left(\frac{1}{\sqrt{8}}\right)$
4	-3	$4\ G - 12\,\mathrm{Ti}_2(3-2\sqrt{2})$
4	-2	$\pi\ \log\left(2+\sqrt{3}\right) - 2\,G - \frac{\pi^2}{8}$
4	-1	$2\log 3 - \frac{2}{3}\,G + 2\,\mathrm{Ti}_2\left(3-2\,\sqrt{2}\right) - \sqrt{8}\ \arctan\left(\frac{1}{\sqrt{8}}\right)$
4	1	$\frac{2}{5} - \frac{G}{10} + \frac{3}{10}\ \mathrm{Ti}_2\left(3-2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10}\ \arctan\left(\frac{1}{\sqrt{8}}\right)$
5	-3	$-\frac{110}{9}\,G - 10\ \log\left(2-\sqrt{3}\right)\ \theta - \frac{1}{8}\,\pi^2 + 5\ \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2}\,\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right)$ $-10\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta+\frac{1}{3}\,\pi\right) + 10\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta-\frac{1}{6}\,\pi\right)$ $+\frac{10}{3}\,\mathrm{Cl}_2\left(\theta+\frac{1}{6}\,\pi\right) + \frac{20}{3}\,\mathrm{Cl}_2\left(\theta+\frac{4}{3}\,\pi\right) - \frac{10}{3}\,\mathrm{Cl}_2\left(\theta+\frac{5}{3}\,\pi\right) - \frac{20}{3}\,\mathrm{Cl}_2\left(\theta+\frac{11}{6}\,\pi\right)$
5	-2	$\frac{8}{3}\,B_5(-6) - \frac{1}{3}\,B_5(-4) + \frac{5}{2}\,\pi\log(3) + 10\,\mathrm{Ti}_2\left(\frac{1}{3}\right) - 10\,G$
5	-1	$-\frac{110}{27}\,G + \frac{10}{3}\ \log\left(2-\sqrt{3}\right)\ \theta + \frac{1}{48}\,\pi^2 + 5\ \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2}\,\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right)$ $+\frac{10}{3}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta+\frac{1}{3}\,\pi\right) - \frac{10}{3}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta-\frac{1}{6}\,\pi\right)$ $-\frac{10}{9}\,\mathrm{Cl}_2\left(\theta+\frac{1}{6}\,\pi\right) + \frac{20}{3}\,\mathrm{Cl}_2\left(\theta+\frac{4}{3}\,\pi\right) - \frac{10}{3}\,\mathrm{Cl}_2\left(\theta+\frac{5}{3}\,\pi\right) + \frac{20}{9}\,\mathrm{Cl}_2\left(\theta+\frac{11}{6}\,\pi\right)$
5	1	$-\frac{77}{81}\,G + \frac{7}{9}\ \log\left(2-\sqrt{3}\right)\ \theta + \frac{1}{360}\,\pi^2 + \frac{1}{6}\,\sqrt{5} + \frac{10}{3}\ \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{4}{3}\,\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) +$ $\frac{7}{9}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta+\frac{1}{3}\,\pi\right) - \frac{7}{9}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta-\frac{1}{6}\,\pi\right)$ $-\frac{7}{27}\,\mathrm{Cl}_2\left(\theta+\frac{1}{6}\,\pi\right) - \frac{14}{27}\,\mathrm{Cl}_2\left(\theta+\frac{4}{3}\,\pi\right) + \frac{7}{27}\,\mathrm{Cl}_2\left(\theta+\frac{5}{3}\,\pi\right) + \frac{14}{27}\,\mathrm{Cl}_2\left(\theta+\frac{11}{6}\,\pi\right)$

Here *G* is Catalan; Cl is Clausen function; Ti is Lewin function; and θ = arctan((16-3*sqrt(15))/11).

Recent evaluations of box integrals, continued

n	s	$\Delta_n(s)$
2	-5	$\frac{4}{3} + \frac{8}{9}\sqrt{2}$
2	-1	$\frac{4}{3} - \frac{4}{3}\sqrt{2} + 4\log(1+\sqrt{2})$
2	1	$\frac{2}{15} + \frac{1}{15}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2})$
3	-7	$\frac{4}{5} - \frac{16\sqrt{2}}{15} + \frac{2\sqrt{3}}{5} + \frac{\pi}{15}$
3	−2	$2\pi - 12\,G + 12\,\mathrm{Ti}_2\left(3-2\sqrt{2}\right) + 6\pi\log\left(1+\sqrt{2}\right) + 2\log 2 - \frac{5}{2}\log 3 - 8\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)$
3	-1	$\frac{2}{5} - \frac{2}{3}\pi + \frac{2}{5}\sqrt{2} - \frac{4}{5}\sqrt{3} + 2\log\left(1+\sqrt{2}\right) + 12\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) - 4\log\left(2+\sqrt{3}\right)$
3	1	$-\frac{118}{21} - \frac{2}{3}\pi + \frac{34}{21}\sqrt{2} - \frac{4}{7}\sqrt{3} + 2\log\left(1+\sqrt{2}\right) + 8\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$
3	3	$-\frac{1}{105} - \frac{2}{105}\pi + \frac{73}{840}\sqrt{2} + \frac{1}{35}\sqrt{3} + \frac{3}{56}\log\left(1+\sqrt{2}\right) + \frac{13}{35}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$

Recent evaluations of box integrals, continued

n	s	$\Delta_n(s)$
4	-10	$-\frac{8}{45}\pi\sqrt{3} - \frac{20}{9}\pi\log(2) + \frac{4}{9}\pi^2 + \frac{4}{3}\log(2) + \frac{1}{15}\log(3) + \frac{8}{3}\mathrm{Ti}_2(3-2\sqrt{2})$
4	-9	$\frac{16}{5}\pi\sqrt{3} - \frac{32}{3}\pi\log(2) - \frac{2}{3}\pi^2 + \frac{16}{5}\pi + 8\sqrt{2}\arctan(2\sqrt{2}) - 24\log(2) + \frac{2}{5}\log(3)$
		$+12\pi\log(\sqrt{2}-1) - 64\mathrm{Ti}_2(3-2\sqrt{2}) + \frac{160}{3}G$
4	-3	$-\frac{128}{15} + \frac{1}{63}\pi - 8\log(1+\sqrt{2}) - 32\log(1+\sqrt{3}) + 16\log(2) + 20\log(3)$
		$-\frac{8}{5}\sqrt{2} + \frac{32}{5}\sqrt{3} - 32\sqrt{2}\arctan\left(\frac{1}{\sqrt{8}}\right) - 96\mathrm{Ti}_2(3-2\sqrt{2}) + 32G$
4	-2	$-\frac{16}{15}\pi\sqrt{3} + \frac{16}{3}\pi\log(1+\sqrt{3}) - \frac{8}{3}\pi\log(2) + 4\pi\log(\sqrt{2}+1) - \frac{2}{3}\pi^2 + \frac{4}{5}\pi$
		$+\frac{8}{5}\sqrt{2}\arctan(2\sqrt{2}) + \frac{2}{5}\log(3) + 8\mathrm{Ti}_2(3-2\sqrt{2}) - \frac{40}{3}G$
4	-1	$\frac{704}{195} - \frac{8}{39}\pi - \frac{100}{13}\log(3) + \frac{120}{13}\log(2) - \frac{8}{65}\sqrt{2} + \frac{128}{65}\sqrt{3}$
		$-\frac{140}{13}\log(1+\sqrt{2}) - \frac{32}{13}\log(1+\sqrt{3}) + \frac{160}{13}\sqrt{2}\arctan\left(\frac{1}{\sqrt{8}}\right) + \frac{48}{13}G$
4	1	$-\frac{23}{135} - \frac{16}{315}\pi - \frac{52}{105}\log(2) + \frac{197}{420}\log(3) + \frac{73}{630}\sqrt{2} + \frac{8}{105}\sqrt{3}$
		$+\frac{1}{14}\log(1+\sqrt{2}) + \frac{104}{105}\log(1+\sqrt{3}) - \frac{68}{105}\sqrt{2}\arctan\left(\frac{1}{\sqrt{8}}\right) - \frac{4}{15}G + \frac{4}{5}\mathrm{Ti}_2(3-2\sqrt{2})$
5	-3	$-\frac{12304}{63} - \frac{512}{21}\sqrt{2} + \frac{576}{7}\sqrt{3} + \frac{800}{21}\sqrt{5} - \frac{320}{3}B_2(3) + \frac{448}{3}B_2(5)$
		$-320B_3(1) + 960B_3(3) - \frac{1792}{3}B_3(5) - 160B_4(-1) + \frac{4400}{3}B_4(1) - \frac{20720}{9}B_4(3)$
		$+896B_4(5) + 32B_5(-3) + \frac{800}{3}B_5(-1) - 1488B_5(1) + \frac{14336}{9}B_5(3) - 448B_5(5)$
5	-1	$\frac{16388}{189} + \frac{1024}{189}\sqrt{2} - \frac{192}{7}\sqrt{3} - \frac{4000}{189}\sqrt{5} + \frac{64}{3}B_2(5) - \frac{192}{7}B_2(7) + \frac{320}{3}B_3(3)$
		$-256B_3(5) + \frac{960}{7}B_3(7) + 160B_4(1) - \frac{6160}{9}B_4(3) + 784B_4(5) - \frac{1760}{7}B_4(7)$
		$+32B_5(-1) - 400B_5(1) + \frac{8192}{9}B_5(3) - 672B_5(5) + \frac{1056}{7}B_5(7)$
5	1	$-\frac{1279}{567}G - \frac{4}{189}\pi + \frac{4}{315}\pi^2 - \frac{449}{3465} + \frac{3239}{62370}\sqrt{2} + \frac{568}{3465}\sqrt{3} - \frac{380}{6237}\sqrt{5}$
		$+\frac{295}{252}\log(3) + \frac{1}{54}\log(1+\sqrt{2}) + \frac{20}{63}\log(2+\sqrt{3}) + \frac{64}{189}\log\left(\frac{1+\sqrt{5}}{2}\right)$
		$-\frac{73}{63}\sqrt{2}\arctan\left(\frac{1}{\sqrt{8}}\right) - \frac{8}{21}\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) + \frac{104}{63}\log(2-\sqrt{3})\,\theta$
		$+\frac{5}{7}\mathrm{Ti}_2(3-2\sqrt{2}) + \frac{104}{63}\mathrm{Cl}_2\left(\frac{1}{3}\theta+\frac{1}{3}\pi\right) - \frac{104}{63}\mathrm{Cl}_2\left(\frac{1}{3}\theta-\frac{1}{6}\pi\right)$
		$-\frac{104}{189}\mathrm{Cl}_2\left(\theta+\frac{1}{6}\pi\right) - \frac{208}{189}\mathrm{Cl}_2\left(\theta+\frac{4}{3}\pi\right) + \frac{104}{189}\mathrm{Cl}_2\left(\theta+\frac{5}{3}\pi\right) + \frac{208}{189}\mathrm{Cl}_2\left(\theta+\frac{11}{6}\pi\right)$

Elliptic Integrals

Recent research in integrals of elliptic functions have revealed hundreds of heretofore unknown identities, for instance:

$$
\begin{aligned}
-2\int_0^1 xK(x)\,\mathrm{d}x + 3\int_0^1 xE(x)\,\mathrm{d}x &= 0 \\
2\int_0^1 K^2(x)\,\mathrm{d}x - 4\int_0^1 K(x)E(x)\,\mathrm{d}x + 3\int_0^1 E^2(x)\,\mathrm{d}x - \int_0^1 K'(x)E'(x)\,\mathrm{d}x &= 0 \\
-2\int_0^1 K^3(x)K'(x)E'(x)\,\mathrm{d}x + \int_0^1 E(x)K'^3(x)E'(x)\,\mathrm{d}x &= 0
\end{aligned}
$$

These studies involved computing thousands of individual definite integrals, each to at least 1600-digit precision, then searching for relations among them using PSLQ.

Summary

- The emerging "experimental" methodology in mathematics and mathematical physics often requires hundreds or even thousands of digits of precision.
- High-precision evaluation of integrals, followed by constant-recognition techniques, has been a particularly fruitful area of recent research, with many new results in pure math and mathematical physics.
- There is a critical need to develop faster techniques for high-precision numerical integration in multiple dimensions.